CSCI 3210
Computational Game Theory

Congestion Games

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News from AAMAS 2024

BEST PAPER AWARDS

Amongst the many excellent submission received, the conference will honour two of the full papers in the main track with awards: the *Best Paper Award* (for which all papers are eligible), and the *Pragnesh Jay Modi Best Student Paper Award* (for a paper with a principal author who is a student).

The three papers listed below arefi nalists for the *Best Paper Award*:

- Yaoxin Ge, Yao Zhang, Dengji Zhao, Zhihao Gavin Tang, Hu Fu and Pinyan Lu. Incentives for Early Arrival in Cooperative Games.
- Evan Albers, Mohammad Irfan and Matthew Bosch. Beliefs, Shocks, and the Emergence of Roles in Asset Markets: An Agent-Based Modeling Approach.
- Grant Forbes, Nitish Gupta, Leonardo Villalobos-Arias, Colin Potts, Arnav Jhala and David Roberts. Potential-Based Reward Shaping for Intrinsic Motivation.

The three papers listed below arefi nalists for the *Pragnesh Jay Modi Best Student Paper Award*:

- Junqi Jiang, Francesco Leofante, Antonio Rago and

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Very first slide of this course

Nash equilibrium (?) and its inefficiency

Braess's paradox

Reading: Ch 8 of EK

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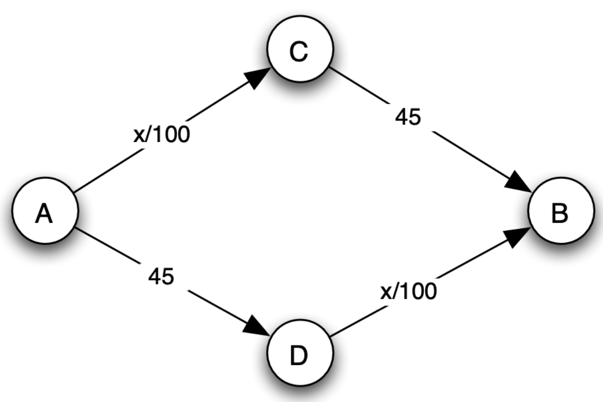
Total # of cars = 4000



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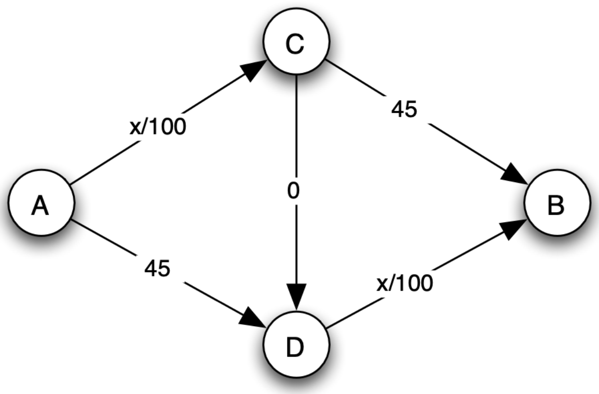

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Computational game theory

- How to model strategic interactions?
- How to compute solutions like NE and CE?
  - Pure vs. mixed NE
- How worse off is NE compared to socially best?

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Congestion game

Terminology	Example
Players	Cars
Resources	Edges in a road network
An action/pure strategy of a player = a subset of resources	A path in a road network
Cost of a resource $\propto$ # of players selecting it	Cost of an edge = # of cars using an edge
Cost faced by a player = Total cost of resources used	Total cost of the path taken by the player

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Can you define a Nash equilibrium?

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A Class of Games Possessing Pure-Strategy Nash Equilibria

By ROBERT W. ROSENTHAL[1])

1973

Abstract: A class of noncooperative games (of interest in certain applications) is described. Each game in the class is shown to possess at least one Nash equilibrium in pure strategies.

1. Description

There are n players ($i = 1, \ldots, n$) and t primary factors ($k = 1, \ldots, t$). The i^{th} player's ($i = 1, \ldots, n$) set of pure strategies contains s_i elements ($r_i = 1, \ldots, s_i$). The r_i^{th} pure strategy may be viewed as the selection of a particular subset of the primary factors. The cost to i of playing the r_i^{th} pure strategy is the sum of the costs of each of the primary factors he selects. The individual factor costs c_k (identical for each player) are functions of x_k, the number of people selecting the k^{th} factor, only. Thus, the cost to player i, if the strategy combination $(r_1, \ldots, r_n)$ is selected, is $\pi_i(r_1, \ldots, r_n) = \sum_{k \in r_i} c_k(x_k(r_1, \ldots, r_n))$. A Nash equilibrium in pure strategies is a pure-strategy combination $(r_1^*, \ldots, r_n^*)$ satisfying

$$\pi_i(r_1^*, \ldots, r_n^*) \leq \pi_i(r_1^*, \ldots, r_{i-1}^*, r_i, r_{i+1}^*, \ldots, r_n^*) \quad r_i = 1, \ldots, s_i;\ i = 1, \ldots, n.$$

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GAMES AND ECONOMIC BEHAVIOR **14,** 124–143 (1996)
ARTICLE NO. 0044

Potential Games

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Received January 19, 1994

We define and discuss several notions of potential functions for games in strategic form. We characterize games that have a potential function, and we present a variety of applications. *Journal of Economic Literature* Classification Numbers: C72, C73.

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Potential game

For any player:

the difference in the payoffs for two strategy profiles

is proportional to

the difference in the **potential function**.

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What is a potential function?

- Global function: irrespective of players
  - Cannot be payoff function in general
- Strategy profile → real number
- Not all games are potential games (meaning they don't have a potential function)

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Computational significance

- Best response dynamics converges into a PSNE
- Running time is pseudopolynomial

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So, which games are potential games?

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THEOREM 3.1. *Every congestion game is a potential game.*

Proof. Let Γ be the congestion game defined by the parameters N, M, $(\Sigma^i)_{i \in N}$, $(c_j)_{j \in M}$.
For each $A \in \Sigma$ define

$$P(A) = \sum_{j \in \cup_{i=1}^{n} A^i} \left(\sum_{l=1}^{\sigma_j(A)} c_j(l) \right). \quad (3.2)$$

Potential function (strategy profile A):

for each resource *j* used in strategy profile A:
for *i* = 1 to # of players using resource *j*:
accumulate cost of *j* when *i* players use *j*

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Theorem 3.2. Every finite potential game is isomorphic to a congestion game.

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Algorithms

- Pseudopolynomial-time best-response dynamics
- Symmetric networks: Polynomial-time network flow algorithm (Fabrikant et al., 2004)

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Weighted Congestion Games

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Weight or demand

- Each player has a weight
- Cost of a resource = function of the sum of the weights of the players using the resource
- Cost faced by a player: same as before

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Complexity

PSNE existence is NP-complete for weighted congestion games, even for constant number of players (Dunkel and Schulz, 2008)

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k-Dimensional Congestion Games (*k*-DCGs)

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Equilibria Computation in Multidimensional Congestion Games: CSP and Learning Dynamics Approaches

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- Players have *k*-dimensional demand vectors: width, length, weight, etc. of vehicles
- The cost of a resource is a real-valued function of the aggregate demand (sum of the demand vectors of the agents selecting it)

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CSP: Key idea

- In congestion games, the identify of the players doesn't affect a player's payoff as long as we know their choices of actions.
- So, what if we explore the "configuration space" (space of aggregated demands) instead strategy profiles?

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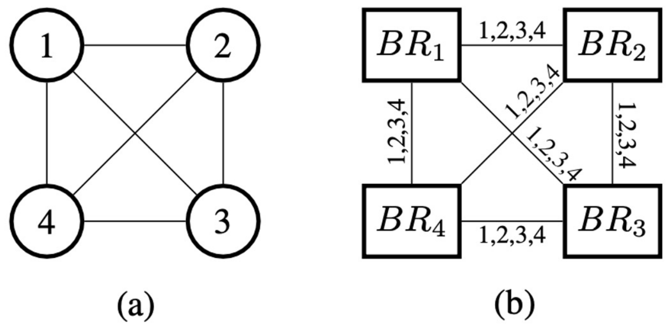
Usual CSP for games



Primal CSP

Dual CSP

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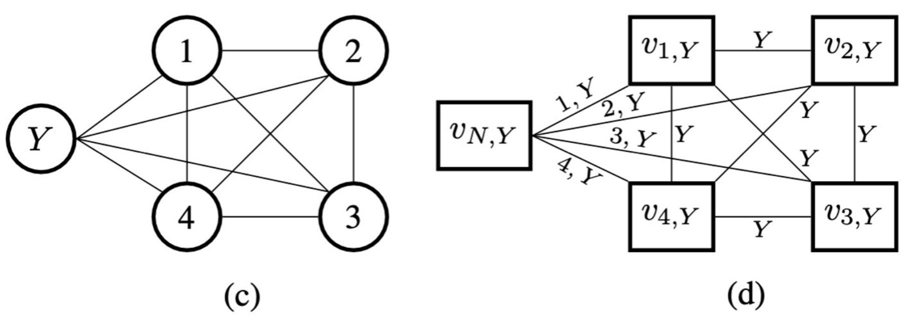
New CSP for congestion games



Primal CSP

Dual CSP

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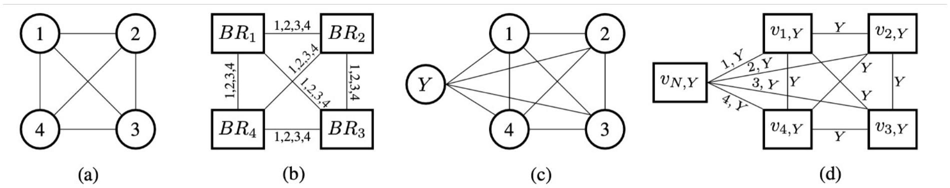

Compare (b) and (d): Decoupling of players' strategies in (d)

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CSP-based algorithm

<u>Procedure 1: Compute the domains of dual variables</u> $v_{i,Y}$:

- What is player *i*'s best response to a given configuration **y**?

- Next, we search for a PSNE without computing the very expensive domain of $v_{N,Y}$ (strategy profiles consistent with **y**).

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CSP-based algorithm

Procedure 2: Search for a PSNE:

Given a configuration **y**, we obtain a PSNE under it when:
(1) Each player plays their best response to **y**.
(2) The resulting aggregate demand is **y**.

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CSP-based algorithm

Solution strategy: dynamic programming:

- Sequentially consider the players' best responses to **y**.
- $T_i(\mathbf{y}')$ = 1 iff ∃ $\mathbf{y}''$ s.t. $T_{i-1}(\mathbf{y}'')$ = 1, and we can go from $\mathbf{y}''$ to $\mathbf{y}'$ by considering some best response of player *i* to **y**.

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Results

Table 1: Our main results on k-dimensional congestion games (k-DCGs), k-class congestion games (k-CCGs), and variants. Notation: NPC $\equiv$ NP-Complete, n = # players, m = # resources, p = max # strategies, $\mathbf{d}_i$ = player i's demand vector, $\mathbf{d}_N = \sum_i \mathbf{d}_i$, $w_{\max} = \max_j \mathbf{d}_{N_j}$, $\check{n}$ = max # players selecting a resource in a binary k-DCG, or max # players of a type in a k-DCG with player types, $l(i)$ = nonzero-element index in $\mathbf{d}_i$ for k-CCG, $a_{\max}$, $b_{\max}$, and $\mathbf{z}$ are cost parameters.
† We give approximation algorithms for (α, β)-PSNE, which always exists. ‡ Klimm and Schütz [2022].

	Problem	PSNE	Time Complexity to Determine or Compute PNSE
CSP Framewrk	General Cost k-DCG	NPC†	$\mathcal{O}\left((w_{\max})^{km}(nkp^2m^2 + nkmp(w_{\max})^{km})\right)$
	Subclass: Binary k-DCG	NPC	$\mathcal{O}\left(\check{n}^{km}(nkp^2m^2 + \min\{nkmp\check{n}^{km}, n^{km+1}p\})\right)$
	Subclass: k-CCG	NPC	$\mathcal{O}\left((w_{\max})^{km}(np^2m^2 + nkpm(w_{\max})^{m})\right)$
	Subclass: k-DCG with player types	NPC	$\mathcal{O}\left((\check{n})^{\tau m}(np^2m^2 + n\tau pm(\check{n})^{m}) + \tau nk\right)$
Learning Dynamics	Linear Cost k-DCG	Always‡	$\mathcal{O}\left(nkpm^2 \times n^2m(a_{\max} + b_{\max})\frac{\max_i [\mathbf{z}\cdot\mathbf{d}_i]^2}{\min_i [\mathbf{z}\cdot\mathbf{d}_i]}\right)$
	Linear Subclass: Binary k-DCG	Always	$\mathcal{O}\left(nkpm^2 \times n^2m(a_{\max} + b_{\max})\left(k \max_j z_j\right)^2\right)$
	Linear Subclass: k-CCG	Always	$\mathcal{O}\left(nkpm^2 \times n^2m(a_{\max} + b_{\max})\frac{\max_j z_j^2}{\min_j z_j}\frac{\max_i d_{i,l(i)}^2}{\min_i d_{i,l(i)}}\right)$
	Exponential Cost k-DCG	Always‡	$\mathcal{O}\left(nkpm^2 \times \frac{e}{e-1}\left(m\exp(\mathbf{z}\cdot\mathbf{d}_N)a_{\max} + nmb_{\max}\right)\right)$
Struct-tured	Ordered $\mathbf{d}_i$'s, nondec. cost, singleton strt.	Always	$\mathcal{O}(n \log n + nmk)$
	Ordered $\mathbf{d}_i$'s, nondec. cost, shared strt.	Always	$\mathcal{O}(n \log n + npmk)$
	Structured cost, singleton strt.	Always	$\mathcal{O}(n \log n + nmk)$

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